The Calderón-Zygmund Decomposition

Let $f \in L^1(\mathbb{R}^n)$, $\alpha > 0$, and $\mathcal{D}$ a dyadic grid on $\mathbb{R}^n$. Then there exist functions g and r on $\mathbb{R}^n$ such that:

1. $f = g + r$
2. $\boxed{\|g\|_{L^1} \leq \|f\|_{L^1}}$ and $\boxed{\|g\|_\infty \leq 2^n \alpha}$
3. $\boxed{r = \sum_{j=1}^{\infty} r_j}$ where each r_j is supported in a dyadic cube Q_j, and the cubes $\{Q_j\}$ are pairwise disjoint (disjoint interiors)
4. $\boxed{\int_{Q_j} r_j = 0}$
5. $\boxed{\langle |r_j| \rangle_{Q_j} \leq 2^{n+1} \alpha}$
6. $\boxed{\sum_j |Q_j| \leq \frac{1}{\alpha} \|f\|_{L^1}}$

This is called the Calderón-Zygmund decomposition of f at height α.

→ The function g is called the good function, since it is both bounded & integrable, hence belongs to all L^p $(1 \leq p \leq \infty)$. More importantly:

$$\|g\|_{L^p} \leq \|g\|_{L^1}^{1/p} \|g\|_\infty^{1/p'} \leq \|f\|_{L^1}^{1/p} (2^n \alpha)^{1/p'}$$

(L^p-norms of g are controlled by the L^1-norm of f!)

→ The function r is often called the "bad function", as in a way it contains the singular part of f, but it is carefully chosen to have mean 0.
From 5 & 6, it follows that r is integrable:

$$\int_{\mathbb{R}^n} |r(x)|\,dx = \sum_j \int_{Q_j} |r_j(x)| = \sum_j \langle |r_j| \rangle_{Q_j} |Q_j| \underset{5}{\leq} 2^{n+1}\alpha \sum_j |Q_j| \underset{6}{\leq} 2^{n+1} \|f\|_{L^1}$$

$$\|r\|_{L^1} \leq 2^{n+1} \|f\|_{L^1}.$$

→ Property 6 is used frequently to obtain sparse domination (via bilinear forms) → later

Proof: We have: $f \in L^1(\mathbb{R}^n)$, $\alpha > 0$ & $\mathcal{D}$. Pick $k_0 \in \mathbb{Z}$ such that $\frac{1}{\alpha}\|f\|_1 \leq 2^{nk_0}$.
Then every dyadic cube $\tilde{Q} \in \mathcal{D}_{k_0}$ i.e. $\ell(\tilde{Q}) = 2^{k_0}$, satisfies:

$$\boxed{\frac{1}{\alpha}\|f\|_{L^1} \leq |\tilde{Q}|} \quad \forall \tilde{Q} \in \mathcal{D}_{k_0}$$

In each $\tilde{Q}$: select the maximal subcubes $Q \subset \tilde{Q}$ s.t. $\boxed{\langle |f| \rangle_Q > \alpha} \rightarrow$ "Stopping condition"
and let $\mathcal{S} := \{Q_j\}_j$ be the collection of all these cubes.

→ Remark: $\tilde{Q}$ itself is never chosen: $\langle |f| \rangle_{\tilde{Q}} \leq \frac{1}{|\tilde{Q}|}\|f\|_1 \leq \alpha$.

→ How the selection goes: for each $\tilde{Q}$, consider its dyadic children $Q \in \tilde{Q}_{(1)}$: select a child if $\langle |f| \rangle_Q > \alpha$. For each remaining child (not selected), consider its children $P \in Q_{(1)}$, and select P if $\langle |f| \rangle_P > \alpha$ ...

→ The resulting cubes $\{Q_j\}$ are indeed disjoint. These will be our cubes for r:

Define:

$$\boxed{r_j := (f - \langle f \rangle_{Q_j}) \mathbb{1}_{Q_j}} \quad \forall j$$

and

$$\boxed{r := \sum r_j = \sum_j (f - \langle f \rangle_{Q_j}) \mathbb{1}_{Q_j}} \quad \boxed{g := f - r} = f - \sum_j (f - \langle f \rangle_{Q_j}) \mathbb{1}_{Q_j}$$

1 ✓ 3 ✓ 4 ✓

For any selected cube $Q_j \in \mathcal{S}$, its parent $\hat{Q}_j$ was not selected, so $\langle |f| \rangle_{\hat{Q}_j} \leq \alpha$

$$\Rightarrow \langle |f| \rangle_{Q_j} = \frac{1}{|Q_j|}\int_{Q_j} |f| \leq \frac{2^n}{|\hat{Q}_j|}\int_{\hat{Q}_j} |f| = 2^n \langle |f| \rangle_{\hat{Q}_j} \leq 2^n \alpha \qquad \langle |f| \rangle_{Q_j} \leq 2^n \alpha$$

$$\Rightarrow \langle |r_j| \rangle_{Q_j} = \frac{1}{|Q_j|}\int_{Q_j} |f - \langle f \rangle_{Q_j}| \leq 2 \langle |f| \rangle_{Q_j} \leq 2^{n+1}\alpha \Rightarrow 5 ✓$$

To prove 6:

$$\sum_j |Q_j| < \frac{1}{\alpha}\sum_j \int_{Q_j} |f(x)|\,dx = \frac{1}{\alpha}\int_{\cup Q_j} |f| \leq \frac{1}{\alpha}\|f\|_{L^1} \quad //$$

$$\langle |f| \rangle_{Q_j} > \alpha \Rightarrow |Q_j| < \frac{1}{\alpha}\int_{Q_j} |f|$$

To prove 2: Remark that: $g(x) = \begin{cases} f(x), & x \notin \cup_j Q_j \\ \langle f \rangle_{Q_j}, & x \in Q_j \end{cases}$ (g coincides with f off Q_j and is constant on each Q_j)

$$\Rightarrow \int_{\mathbb{R}^n} |g| = \underbrace{\sum_j \int_{Q_j} |\langle f \rangle_{Q_j}|}_{\sum_j |\int_{Q_j} f| \leq \sum_j \int_{Q_j} |f| = \int_{\cup Q_j} |f|} + \int_{\mathbb{R}^n \setminus \cup Q_j} |f| \leq \int_{\mathbb{R}^n} |f| \Rightarrow \|g\|_1 \leq \|f\|_1.$$

If $x \in (\overline{\cup_j Q_j})^c$, there is a sequence of dyadic cubes $\{P_k\}$ s.t. $x \in P_k$, $\ell(P_k) \xrightarrow[k\to\infty]{} 0$ and no P_k is in $\mathcal{S}$. So $|\langle f \rangle_{P_k}| \leq \langle |f| \rangle_{P_k} \leq \alpha \Rightarrow$ by LDT: $|f(x)| = |\lim_{k\to\infty} \langle f \rangle_{P_k}| \leq \alpha$ holds for a.a. $x \notin \cup_j Q_j$. Since $g = f$ on $\mathbb{R}^n \setminus \cup_j Q_j$:

$$\left.\begin{array}{l} |g| \leq \alpha \text{ a.e. on } \mathbb{R}^n \setminus \cup_j Q_j \\ |g| = |\langle f \rangle_{Q_j}| \leq 2^n \alpha \text{ on each } Q_j \end{array}\right\} \Rightarrow \|g\|_\infty \leq 2^n \alpha \quad //$$

§. Proof of the weak (1,1) inequality for the dyadic square function:

$$|\{x\in\mathbb{R} : S_{\mathcal{D}}f(x)>\alpha\}| \lesssim \frac{1}{\alpha}\|f\|_1.$$

→ Recall: $S_{\mathcal{D}}$ is quasilinear: $S_{\mathcal{D}}(f+g) \le \sqrt{2}(S_{\mathcal{D}}f + S_{\mathcal{D}}g)$

→ CZ decomposition for $f\in L^1(\mathbb{R})$ at level $\alpha>0$: $\boxed{f=g+h}$ → $\boxed{\|g\|_2^2 \le 2\alpha\|f\|_1}$

→ $\boxed{h=\sum_j h_j}$ → $\boxed{\operatorname{supp}(h_j)\subset I_j}$ $I_j\in\mathcal{D}$, pw. disjoint

→ $\boxed{\int_{I_j} h_j = 0}$

→ $\boxed{\sum_j |I_j| \le \frac{1}{\alpha}\|f\|_1}$

$\Rightarrow$ $\alpha < S_{\mathcal{D}}f(x) \Rightarrow \alpha < S_{\mathcal{D}}(g+h)(x) \le \sqrt{2}(S_{\mathcal{D}}g + S_{\mathcal{D}}h)(x)$

$\Rightarrow \alpha/\sqrt{2} < S_{\mathcal{D}}g(x) + S_{\mathcal{D}}h(x)$

$\Rightarrow$ at least one of $S_{\mathcal{D}}g(x)$ & $S_{\mathcal{D}}h(x)$ must be $> \alpha/2\sqrt{2}$

$$\Rightarrow |\{x\in\mathbb{R}: S_{\mathcal{D}}f(x)>\alpha\}| \le \underbrace{|\{x : S_{\mathcal{D}}g(x) > \alpha/2\sqrt{2}\}|}_{\lesssim 1/\alpha \|f\|_1} + \underbrace{|\{x : S_{\mathcal{D}}h(x) > \alpha/2\sqrt{2}\}|}_{\lesssim 1/\alpha \|f\|_1}$$

Chebyshev:

$$|\{x : S_{\mathcal{D}}g(x) > \alpha/2\sqrt{2}\}| \le \frac{1}{(\alpha/2\sqrt{2})^2}\|g\|_2^2 \lesssim \frac{1}{\alpha^2}\,\alpha\|f\|_1 = \frac{1}{\alpha}\|f\|_1.$$

$$\mu\{|f(x)|>t\} \le \frac{1}{t^p}\int_{|f|\ge t} |f|^p\,d\mu$$

Property of the square function: $\boxed{\left.\begin{array}{l}\operatorname{supp}(h)\subset I\in\mathcal{D}\\ \int_I h = 0\end{array}\right\} \Rightarrow \operatorname{supp}(S_{\mathcal{D}}h)\subset I}$

$$\left.\begin{array}{l}\operatorname{supp}(h)\subset I\\ \int_I h=0\end{array}\right\} \Rightarrow (h,h_J) = \int_I h h_J = \begin{cases} J\subset I: & (h,h_J)\\ J\supsetneq I: & h_J(I)\int_I h = \underline{\underline{0}}\end{cases}$$

$$\Rightarrow S_{\mathcal{D}}^2 h(x) = \sum_{J\subset I} (h,h_J)^2 \frac{\mathbb{1}_J(x)}{|J|} \Rightarrow \text{clearly } 0 \text{ if } x\notin I.$$

$$\Rightarrow S_{\mathcal{D}}^2 h(x) = S_{\mathcal{D}}^2\Big(\sum h_j\Big)(x) = \sum_J \Big|\sum_j (h_j,h_J)\Big|^2 \frac{\mathbb{1}_J(x)}{|J|} = \sum_j \Big(\sum_{I\subset I_j} (h_j,h_I)^2 \frac{\mathbb{1}_I(x)}{|I|}\Big)$$

$$= \sum_j S_{\mathcal{D}}^2 h_j(x)$$

for each j, this is 0 unless $J\subset I_j$
Since the I_j's are disjoint, any one $J\in\mathcal{D}$ can only be contained in at most one I_j

$\Rightarrow \boxed{S_{\mathcal{D}}^2 h = \sum_j S_{\mathcal{D}}^2 h_j}$ & $\operatorname{supp}(S_{\mathcal{D}}h_j)\subset I_j$

$$\Rightarrow S_{\mathcal{D}}^2 h(x) > \frac{\alpha^2}{8} \Rightarrow \sum_j S_{\mathcal{D}}^2 h_j(x)\mathbb{1}_{I_j}(x) > \frac{\alpha^2}{8} \Rightarrow x\in I_j \text{ for some } j \text{ and } S_{\mathcal{D}}h_j(x) > \frac{\alpha}{2\sqrt{2}}$$

$$\Rightarrow |\{x : S_{\mathcal{D}}h(x) > \alpha/2\sqrt{2}\}| = \sum_j |\{x\in I_j : S_{\mathcal{D}}h_j(x) > \alpha/2\sqrt{2}\}| \le \sum_j |I_j| \le \frac{1}{\alpha}\|f\|_1.$$